The 6th Mathematics Conference for Young Researchers 2/16 11:10-11:40

An Introduction to Evolutionary Game Theory

: To Understand the Complex Phenomena

(進化ゲーム理論入門 – 複雑現象理解のために –)

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1. INTRODUCTION

This Talk (本報告)

- **INTRODUCING** to **Game theory** for the beginner. (ゲーム理論を紹介する)

- **DEFINING** game theory mathematically.

(ゲーム理論と数学的に定式化する)

- **PRESENTING** the open problems in this field.

(未解決の問題を紹介する)

Nobel Prize in Economics and Game Theory

- 1994 - J.C. Harsanyi , J.F. Nash and R. Selten (Non-Cooperative Game Theory)
- 2005 - R.J. Aumann and T.C. Schelling (Conflict and Cooperation)
- 2007 - L.Hurwicz, E.S. Maskin and R.B. Myerson (Mechanism Design Theory)
- 2009 - E.Ostrom and O.E. Williamson (Governance : commons, boundaries of the firm)

The Kyoto Prize :2001 – J. Maynard Smith
(Evolutionarily Stable Strategy)

Application

- **Industrial Organization** (産業組織論)
- Player : Firm
- **Cournot**-Nash Duopoly Game (Strategy : Quantity), **Bertrand** Duopoly Game (Strategy: Price)

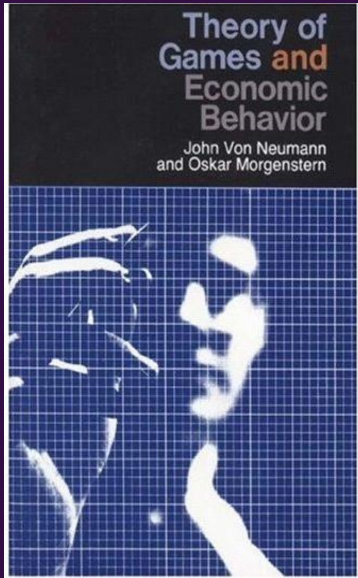
- **Strategic Trade Policy** (戦略的貿易政策)
- Player : Country
- Strategy : Subsidy or no.
- METI (経済産業省) : 「官僚たちの夏」

- **Biology:**

Describe the competition among species.

Ex) Animals (foods), Plants (light), Sex-Ratio game … etc.

J.von Neumann



Theory of Games and Economic Behavior
John Von Neumann and Oskar Morgenstern

1. **A Model of General Economic Equilibrium, RES, 31(1945-46), 1-9.**

→ He proved the existence of situations of equilibrium in mathematical models of market development based on supply and demand by applying Brouwer’s fixed point theory.

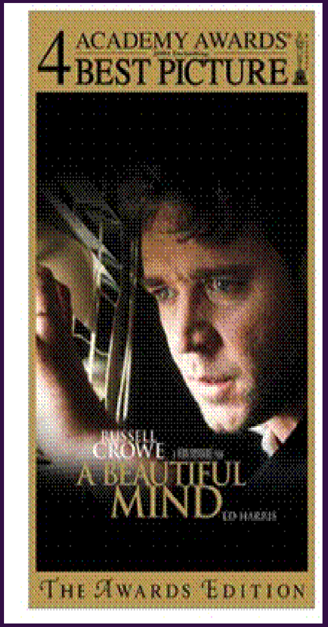
2. **Theory of Game and Economic Behavior (With Oskar Morgenstern) , 1944.**

John Forbes Nash

Nash equilibrium

He shared the 1994 Nobel Prize in Economics with two other game theorists, Reinhard Selten and John Harsanyi.

His most famous work in pure mathematics was the **Nash embedding theorem**, which showed that any abstract Riemannian manifold can be isometrically realized as a submanifold of Euclidean space. He also made contributions to the theory of nonlinear parabolic partial differential equations.



A Beautiful Mind

EXAMPLE
Payoff Matrix is Very Important

- In the end, these applications are to famous games. (これら多くの応用問題は以下の(有名な)ゲームに帰着する)
- **Prisoner’s Dilemma** (囚人のジレンマ) Game･･･ Environmental Problem (環境問題), Cournot Duopoly(複占市場), Public Goods Game (公共財支出)

player 2

		C	D
player1	C	2年,2年	7年,無罪
	D	無罪, 7年	5年,5年

N.E. : (D,D)

- **Coordination Game**・・・Standardize(規格統一)

player 2

player1

	S1	S2
S1	+,+	0,0
S2	0,0	+,+

N.E. : (S1,S1), (S2,S2)

- **Hawk-Dove Game**・・・Struggle between animals (生物種における闘争)

N.E. : Mixed Strategy

player 2

player1

	Hawk	Dove
Hawk	V-C/2, V-C/2	V,0
Dove	0,V	V/2, V/2

C>V

2. NONCOOPERATIVE GAME THEORY

2.1 DEF. Strategic Game

DEF. A strategic game is

(2.1) $$G=\left(N,\{S_i\}_{i\in N},\{f_i\}_{i\in N}\right).$$

where (i) $N=\{1,2,\ldots,n\}$ is the set of **players**, (ii) S_i is the set of **strategies/actions** available to player *i*. All the players' strategies are expressed by $\vec{s}=s_1,\cdots,s_n$. The strategy s_i is called a **pure strategy**. (iii) f_i is a measurable function from the product set $\vec{S}=S_1\times\cdots\times S_n$ to a real number and this is represented by a player *i*'s **utility function**.

ASSUMPTIONS

- Each player knows the details of the game. All the players *1,…, n* choose their strategy simultaneously and independently. After the game, the player *i* obtains a payoff $f_i(\vec{s})$.

Assum. $\forall i$, S_i is a separable complete metric space.

Assum. $\forall i$, $f_i : \vec{s} \to \Re$ is a bounded continuous function.

Assum. The player's purpose maximizes the own utility.

Assum. Common knowledge: All the players know the own utility function and another players' utility function.

2.6 DEF. Mixed Extension

DEF. A **mixed extension** is

(2.2) $$G^* = \left(N, \{Q_i\}_{i\in N}, \{F_i\}_{i\in N}\right).$$

where (i) *N={1,2,…,n}* is the set of players, (ii) Q_i is all the probability distribution on S_i. Q_i is a random variable and it is called a mixed strategy. We assume that the random vector $\vec{q} = q_1, \cdots, q_n$. (iii) F_i is a real valued function on the product set $\vec{Q} = Q_1 \times \cdots \times Q_n$. This is defined as follows:

(2.3) $$F_i(\vec{q}) = \int_{\vec{Q}} f_i(\vec{s})\, d\mu(\vec{s}),$$

where μ is the $\vec{q}\,'s$ distribution. $F_i(\vec{q})$ is an expected payoff function of player *i*. The set of expected payoff function is $\vec{F} = F_1, \cdots, F_n$.

Assumption 2.7, Remark2.8

Assum. The random variable q_i, $i=1,\ldots,n$ is independent.

REM. $q_1,\ldots,q_n$ is independent and each q_k's distribution $\mu_k \in Q_k$. (2.3) is reduced as follows:

(2.3')
$$F_i(\vec{q}) = \int_{\vec{Q}} f_i(\vec{s})d\mu_1(s_1)\cdots\mu_n(s_n).$$

Here, $\overline{Q}$ is all the probability measures on $\vec{S}$ and $\vec{\mu}_1, \vec{\mu}_2$ are the distribution on the set of mixed strategy $\vec{q}^{\,1}, \vec{q}^{\,2}$.

Under $\vec{\mu}_1, \vec{\mu}_2 \in \overline{Q}$, we can transform as follows:

$$\vec{\mu}_\alpha = \alpha\vec{\mu}_1 + (1-\alpha)\vec{\mu}_2, \quad 0 \le \alpha \le 1.$$

We can define the new probability measure

We can understand that $\overline{Q}$ is a convex set. $\vec{Q}$ Is the closed set in $\overline{Q}$.

DEF. Feasible Set, Remark.

DEF. A feasible set *U* on $G^* = \left(N, \{Q_i\}_{i \in N}, \{F_i\}_{i \in N} \right)$ is defined as follows:

$$U = \left\{ \vec{F}\left(\vec{q}\right) \middle| \vec{q} \in \vec{Q} \right\}.$$

REM. The feasible set *U* is a compact set on the separable complete metric space for the continuity of the expected payoff function $\vec{F}$

DEF. Best Response

DEF. : A best response of the player *i'*s strategy $q_i \in Q_i$

For another *n-1* players' strategy sets $q_{-i}=(q_1,\dots,q_{i-1},q_{i+1},\dots,q_n)$ is

$$F_i\left(q_i, q_{-i}\right) = \max_{r_i \in Q_i} F_i\left(r_i, q_{-i}\right)$$

The whole best response for player *i* is $B_i(q_{-i})$ for strategy set q_{-i}.

DEF. Nash eq., Remark.

DEF. A **Nash equilibrium** of a strategic game n-person game *G** is a profile $\overrightarrow{q^*} = \left(q_1^*, \cdots, q_n^*\right)$ with the property that for every player *i=(1,…,n)* we have the best response for another player's strategy set q^*_{-1}.

REM. : The mapping is a point to set mapping from the product set $Q_1 \times \ldots \times Q_{i-1} \times Q_{i+1} \times \ldots \times Q_n$ to set Q_j. It is called a **best response correspondence** for player *i*.

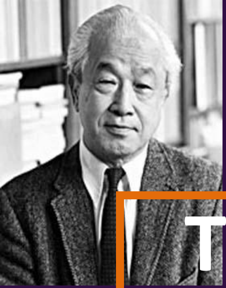
$B(q)=B_1(q_{-1}) \times \ldots \times B_n(q_{-n})$ for the strategy set $\vec{q}$.

Theorem 2.14, 2.15.

TH. The mixed strategy $\overrightarrow{q^*} = \left(q_1^*, \cdots, q_n^*\right)$ on *G** is a Nash equilibrium if and only if

(2.5) $\overrightarrow{q^*} \in B(q^*)$

TH. The strategic game *G** has a Nash equilibrium.



Kakutani’s Fixed Point Theory

TH.: Let *S* be a nonempty, compact convex subset of the separable complete metric space and let $F(.) : S \to S$ be a set-valued mapping for which

(i) For all $x \in S$ the set $F(x)$ is a nonempty set and convex on *S*.

(ii) for all sequence $\{x_\nu\}_{\nu=1}^{\infty}$ and $\{y_\nu\}_{\nu=1}^{\infty}$ such that

$y_v \in F(x_v)$, $v=1,2,\ldots,x_v \to x_O$, $y_v \to y_O$ $(v \to \infty)$,

we have $y_0 \in F(x_O)$.

Then there exists $x^* \in F(.)$ such that $x^* \in F(x^*)$

3. EVOLUTIONARY GAME THEORY

EVOLUTIONARILY STABLE STRATEGY（ESS)

DEF.：Weibull(1995): $x \in \Delta$ is an ***evolutionarily stable strategy (ESS)*** if for every strategy $y \neq x$ there exists some $\overline{\varepsilon_y} \in (0,1)$ such that the following inequality holds for all $\varepsilon \in (0, \overline{\varepsilon_y})$.

$$u[x, \varepsilon y + (1-\varepsilon)x] > u[y, \varepsilon y + (1-\varepsilon)x].$$

INTERPRETATION：incumbent payoff (fitness) is higher

than that of the post-entry strategy

（ESS : ①the solution of the Replicator equation + ② asymptotic stable.）

PROPOSITION

PRO.(Bishop and Cannings (1978)): $x \in \Delta$ is evolutionary stable strategy if and only if it meets these first-order and second-order best-reply :

Nash Eq.

$$(2.4) \quad u(y,x) \le u(x,x), \quad \forall y,$$

$$(2.5) \quad \begin{array}{l} u(y,x) = u(x,x) \\ \Rightarrow u(y,y) < u(x,y), \end{array} \quad \forall y \ne x,$$

Asymptotic Stable Conditon

Bishop, D. T. and Cannings, C : Models of animal conflict, Advances Applied Probability, Vol. **8 (1976)** , pp.616-621.

DEF. 3.3. selection dynamics

DEF.: Let $\pi : \vec{q} \rightarrow \Re$. Then the system

(3.4) $$\overset{\bullet}{\vec{q}} = \pi(\vec{q}),$$

is a selection dynamics if it satisfies, for all $q_i \in Q_i$

(3.5) (i) π is Lipschitz continuous.

(3.6) (ii) $$\sum_{i=1}^{n} \pi_i \left(\vec{q}\right) = 0.$$

(3.7) (iii) $\forall\, q_i \in Q_i$, $qi = 0 \Rightarrow$ $\pi(\vec{q}) \geq 0.$

DEF. 3.4, 3.5 regular, monotonic

DEF.: π yields a regular selection dynamics if (3.5)-(3.7) holds; then the following limits exist and are finite:

(3.8) $$\frac{\pi}{0} \equiv \lim_{q_i \to 0} \frac{\pi}{q_i}.$$

DEF.: π_i is **monotonic** if, for *i,i’* ∊ *N*

(3.9) $$F(q_i, q_{-i}) \geq F(q_{i'}, q_{-i'}) \Rightarrow \frac{\pi_i(\vec{q})}{q_i} \geq \frac{\pi_{i'}(\vec{q}')}{q_{i'}}.$$

⇒A higher payoff is a higher increment in this selection dynamics.

DEF. Replicator eq., Picard-Lindelof theorem

DEF.: If the selection dynamics is monotonic, we can derive the following equation. The selection dynamics π is a **replicator equation**, if

(3.10)

$$\frac{\pi_i\left(\vec{q}\right)}{q_i} = F(q_i, q_{-i}) - \sum_{i=k}^{n} q_k F(q_k, q_{-k}).$$

Th. (Picard-Lindelof theorem) If $X \subset R^k$ is open and the vector field ψ: $X \to R^k$ Lipschitz continuous, then the system (3.4) has a unique solution $\xi(.,x^o): T \to X$ through every state $x^o \in X$. Moreover $\xi(t, x^o)$ is continuous in $t \in T$ and $x^o \in X$.

EXAMPLE

1) Symmetric two person game with two strategies

$$\dot{x} = x(1-x)\{b-(a+b)x\}$$

2

1

	S 1	S 2
S 1	a,a	0,0
S 2	0,0	b,b

Payoff matrix

Classification

- **(I) Non-dilemma**: a > 0, b < 0, ESS : one
- **(II) Prisoner's dilemma** : a < 0, b > 0, ESS :one
- **(III) Coordination** : a>0,b>0, ESS two
- **(IV) Hawk-Dove** : a<0,b < 0, ESS one (mixed strategy)

EXTENTION:

$$\dot{y} = y(1-y)\{a-(a+c)x\}, \dot{x} = x(1-x)\{d-(b+d)y\}.$$

$$\frac{\partial x_i}{\partial t} = D \cdot \nabla^2 x_i + x_i(1-x_i)\left(ax_i - b(1-x_i)\right)$$

4. OPEN PROBLEMS AND FUTURE WORKS

OPEN PROBLEMS

1. Infinite strategy spaces
2. Without Common Knowledge (2.5 Asuum.)
3. Extend a replicator equation (PDE, SDE)
4. As Statistical Mechanics (Many games are played simultaneously)

In progress

ANALYZING the financial market with Evolutionary Game Theory.

[Movie]

Thank You For Your Attention

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This File is available at

http://kikkawa.cyber-ninja.jp/

Next my talk: 2/28 @ Taipei

Text Book

For Detail, See my Website(**Bookguide** **Readinglist**)

Classsic:

[1] Maynard Smith, John **Evolution and the Theory of Games** , Cambridge University Press, 1982/10. 日本語訳

[2] Axelrod, Robert **The Evolution of Cooperation** , Basic Books, 1984/03. 日本語訳

Text Book:

[1] Weibull , Jorgen W. **Evolutionary Game Theory** , MIT Press, 1995/08/14. 日本語訳

[2] Hofbauer, Josef and Sigmund, Karl **Evolutionary Games and Population Dynamics** , Cambridge University Press, 1998/07. 日本語訳

[3] Vega-Redondo , Fernando **Evolution, Games and Economic Behaviour** , Oxford University Press, 1997/01.

[4] Samuelson, Larry **Evolutionary Games and Equilibrium Selection (Mit Press Series on Economic Learning and Social Evolution, 1)** , MIT Press, 1997/04.

For Beginner :

[1] 石原英樹, 金井雅之 **進化的意思決定 (シリーズ意思決定の科学)** , 朝倉書店, 2002/04/05.

[2]大浦宏邦 **社会科学者のための進化ゲーム理論—基礎から応用まで** 書房, 2008/09/25.

- 本研究の一部は，平成20年度採択, 文部科学省 グローバルCOEプログラム**「現象数理学の形成と発展」** 現象数理若手プロジェクト「人間特有の現象に対する学習の影響 - 進化ゲーム理論による分析 -」に関する研究拠点形成費の助成を受けて行われた.