An Introduction to Evolutionary Game Theory
(進化ゲーム理論入門 – 複雑現象理解のために –)

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This Talk (本報告)

- We introduce to **Evolutionary game theory** for the beginner.(進化ゲーム理論を紹介する)

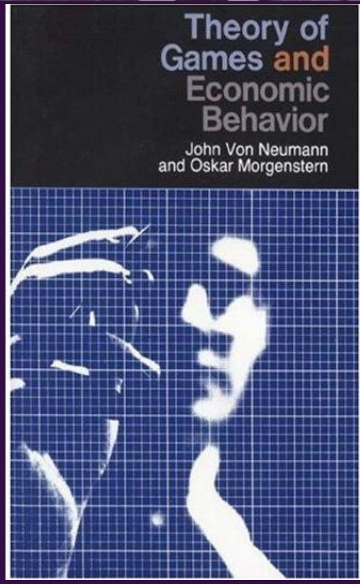
- **What** is evolutionary game theory?
(進化ゲーム理論とは？)
- **Why** do we use evolutionary game theory ?
(なぜ、進化ゲーム理論なのか？)
- What is the **merits** of using evolutionary game theory?
(進化ゲーム理論の特徴とは？)

Self-Introduction : Research

- **Research Field** : Evolutionary Game Theory : Theory and its Applications (進化ゲーム理論とその応用)
- **Theory**: Pay Attention to the Stochastic Aspect in (Evolutionary) Game Theory. (確率的側面に着目)
- **Main Interest** (applications): Economic Phenomena (経済現象). Recently, Finance (最近はファイナンス).

- 「進化ゲーム理論の数理」(A Mathematical Principle of Evolutionary Game Theory)『北海道大学数学講究録』Series #126(2008年1月), pp.173-177. **[HP]**

J.von Neumann



1. **A Model of General Economic Equilibrium, RES, 31(1945-46), 1-9.**

→ He proved the existence of situations of equilibrium in mathematical models of market development based on supply and demand by applying Brouwer's fixed point theory.

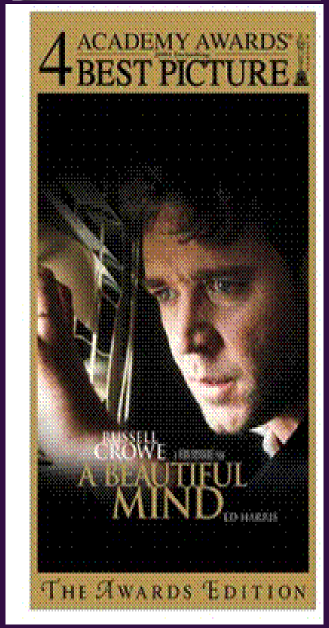
2. **Theory of Game and Economic Behavior (With Oskar Morgenstern) , 1944.**

John Forbes Nash

Nash equilibrium

He shared the 1994 Nobel Prize in Economics with two other game theorists, Reinhard Selten and John Harsanyi.

His most famous work in pure mathematics was the **Nash embedding theorem**, which showed that any abstract Riemannian manifold can be isometrically realized as a submanifold of Euclidean space. He also made contributions to the theory of nonlinear parabolic partial differential equations.



A Beautiful Mind

Nash Equilibrium

DEF. The strategy profile $s^*=(s^*_1,\ldots,s^*_n)$ is a **Nash equilibrium** if

$$f_i(s^*) \geqq f_i(s_i, s^*_{-i}), \quad \forall s_i \in S_i$$

for all players $i(=1,\ldots,n)$, where f is a payoff function and S_i is the set of pure strategy.

→If the player plays the strategy profile s^*, each player has no motivation to change my strategy.

John F. Nash "Equilibrium Points in n-Person Games," PNAS, **36**, (1950) 48-49. **[HP]**

WHAT IS GAME？
(Non-cooperative Game)

There are two interacted players (Player 1, Player 2). (相互関係にある2人の主体がいるとする) If player 1 chooses strategy 1 and player 2 chooses strategy 1, player 1‘s payoff is a , player 2’s payoff b. In this situation, which strategy each players chooses ? (このような状況のもとで、各主体はどのような戦略をとるのか？) ((The game is played only one.)ただしこのゲームは一度限りであるとする))

→This game’s solution is **Nash Equilibrium**.

Player 2

		S1	S2
player1	S1	a,b	0,0
	S2	0,0	c,d

Nash equilibrium depends on the sign of a,b,c,d.

DEF. Strategic Game

DEF. A strategic game

$$G = \left(N, \{S_i\}_{i \in N}, \{f_i\}_{i \in N} \right)$$

consists of

(i) N={1,2,…,n} is the set of players,

(ii) for each player i∈N a nonempty set S_i (the set of actions available to player i)

(iii) for each player i∈N a preference/utility f_i on $\vec{S} = S_1 \times \cdots \times S_n$

Existence of a Nash Equilibrium

Theorem (Nash 1950 [HP]): Every finite strategic-form game has a mixed strategy equilibrium.

Lemma (Kakutani's fixed point theorem):

Let X be a compact convex subset of R^n and let f: X → X be a set-valued function for which

1) For all x ∈ X be the set f(x) is nonempty and convex

2) The graph of f is closed (i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \rightarrow x$, and $y_n \rightarrow y$, we have $y \in f(x)$).

Then there exists $x \in X$ such that $x \in f(x^*)$.

Application

- **Industrial Organization** (産業組織論)
- Player : Firm
- **Cournot**-Nash Duopoly Game (Strategy : Quantity), **Bertrand** Duopoly Game (Strategy: Price)

- **Strategic Trade Policy** (戦略的貿易政策)
- Player : Country
- Strategy : Subsidy or no.
- METI (経済産業省) :「官僚たちの夏」

- **Biology:**

Describe the competition among species.

Ex) Animals (foods), Plants (light), Sex-Ratio game … etc.

EXAMPLE
Payoff Matrix is Very Important

- In the end, these applications are to famous games. (これら多くの応用問題は以下の(有名な)ゲームに帰着する)
- **Prisoner's Dilemma** (囚人のジレンマ) Game･･･ Environmental Problem (環境問題), Cournot Duopoly(複占市場), Public Goods Game (公共財支出)

player 2

player1

	C	D
C	2年,2年	7年,無罪
D	無罪, 7年	5年,5年

N.E. : (D,D)

- **Coordination Game**・・・Standardize(規格統一)

player 2

	S1	S2
S1	+,+	0,0
S2	0,0	+,+

player1

N.E. : (S1,S1), (S1,S1)

- **Hawk-Dove Game**・・・Struggle between animals (生物種における闘争)

N.E. : Mixed Strategy

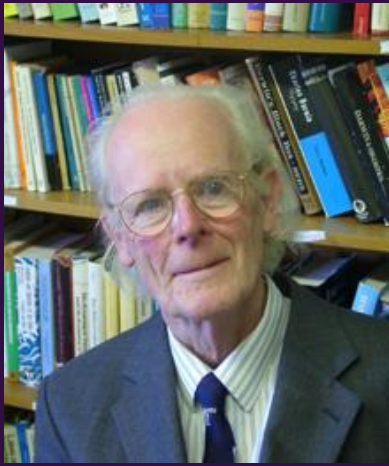
player 2

	Hawk	Dove
Hawk	V-C/2, V-C/2	V,0
Dove	0,V	V/2, V/2

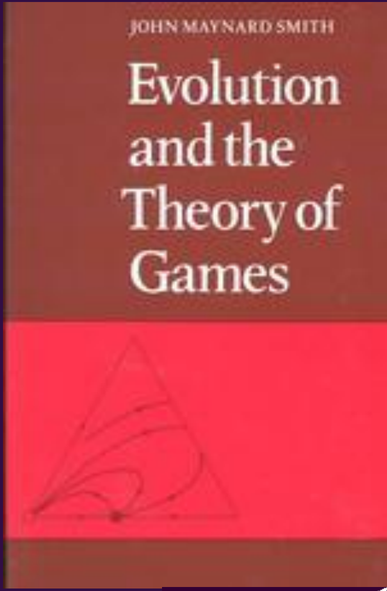
player1

C>V

WHAT IS " EVOLUTIONARY GAME THEORY" ?



In 1973 Maynard Smith formalized a central concept in game theory called the evolutionary stable strategy (ESS), based on a verbal argument by G.R.Price. This area of research culminated in his 1982 book ***Evolution and the Theory of Games***. The Hawk-Dove game is arguably his single most influential game theoretical model.



POINT！: Payoff(利得)→ Fitness(適応度)

EVOLUTIONARILY STABLE STRATEGY（ESS)

DEF.： Weibull(1995): $x \in \Delta$ is an ***evolutionarily stable strategy (ESS)*** if for every strategy $y \neq x$ there exists some $\overline{\varepsilon}_y \in (0,1)$ such that the following inequality holds for all $\varepsilon \in (0, \overline{\varepsilon}_y)$.

$$u[x, \varepsilon y + (1-\varepsilon)x] > u[y, \varepsilon y + (1-\varepsilon)x].$$

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INTERPRETATION：incumbent payoff (fitness) is higher than that of the post-entry strategy

（ESS : ①the solution of the Replicator equation + ② asymptotic stable.）

PROPOSITION

PRO.(Bishop and Cannings (1978)): $\chi \in \Delta$ is evolutionary stable strategy if and only if it meets these first-order and second-order best-reply :

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$$(2.5) \quad \begin{array}{l} u(y,x) = u(x,x) \\ \Rightarrow u(y,y) < u(x,y), \end{array} \quad \forall y \neq x,$$

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Nash Eq.

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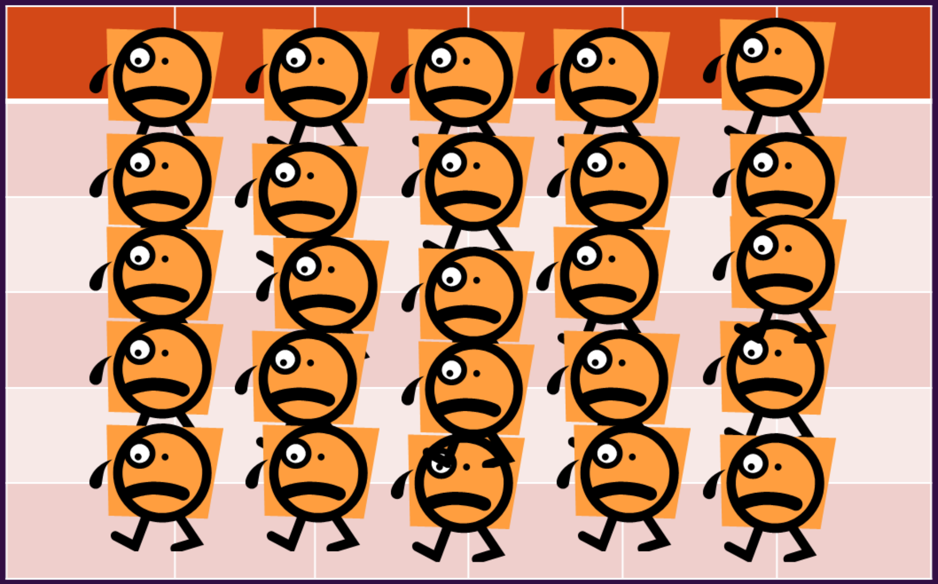
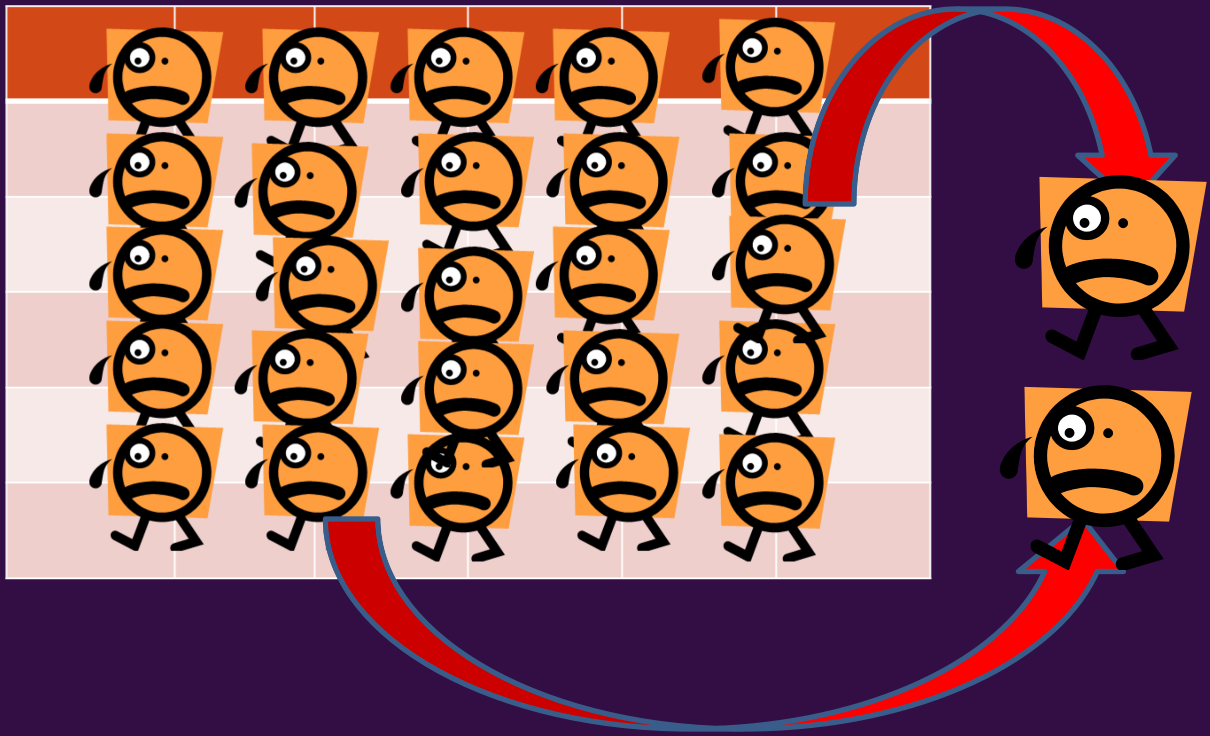
Asymptotic Stable Conditon

Situation (Traditional Evolutionary Game Theory)



Situation (Traditional Evolutionary Game Theory)

At Random (infinitely)



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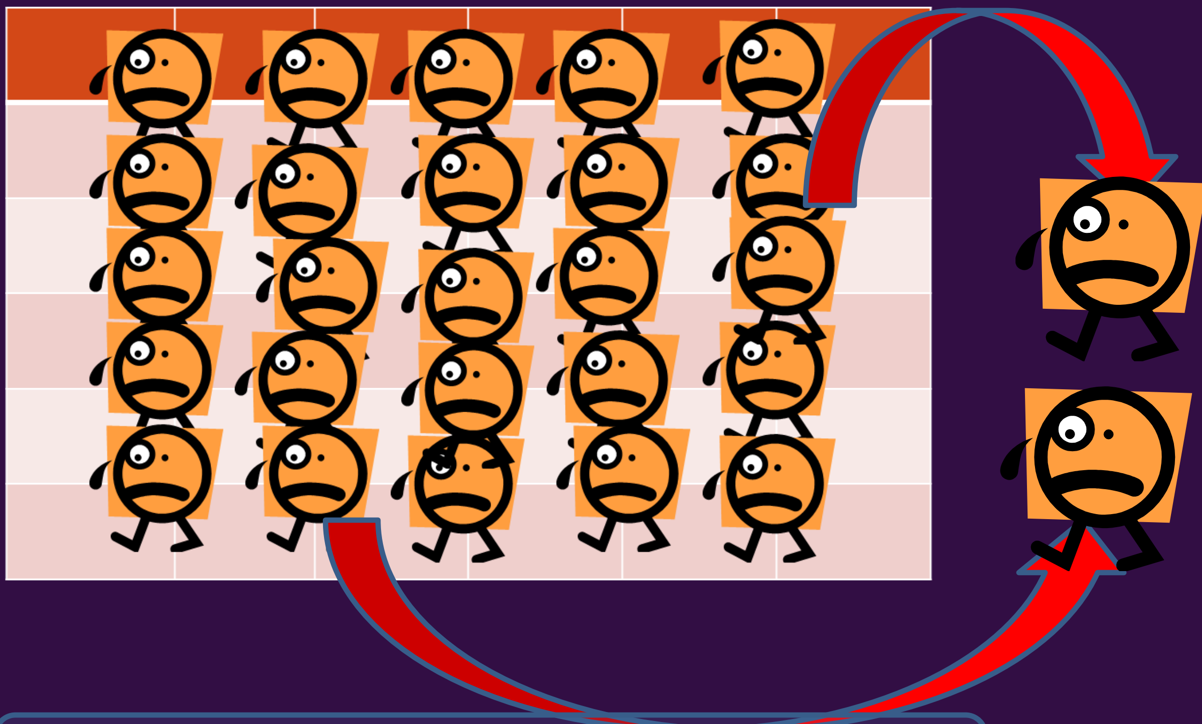
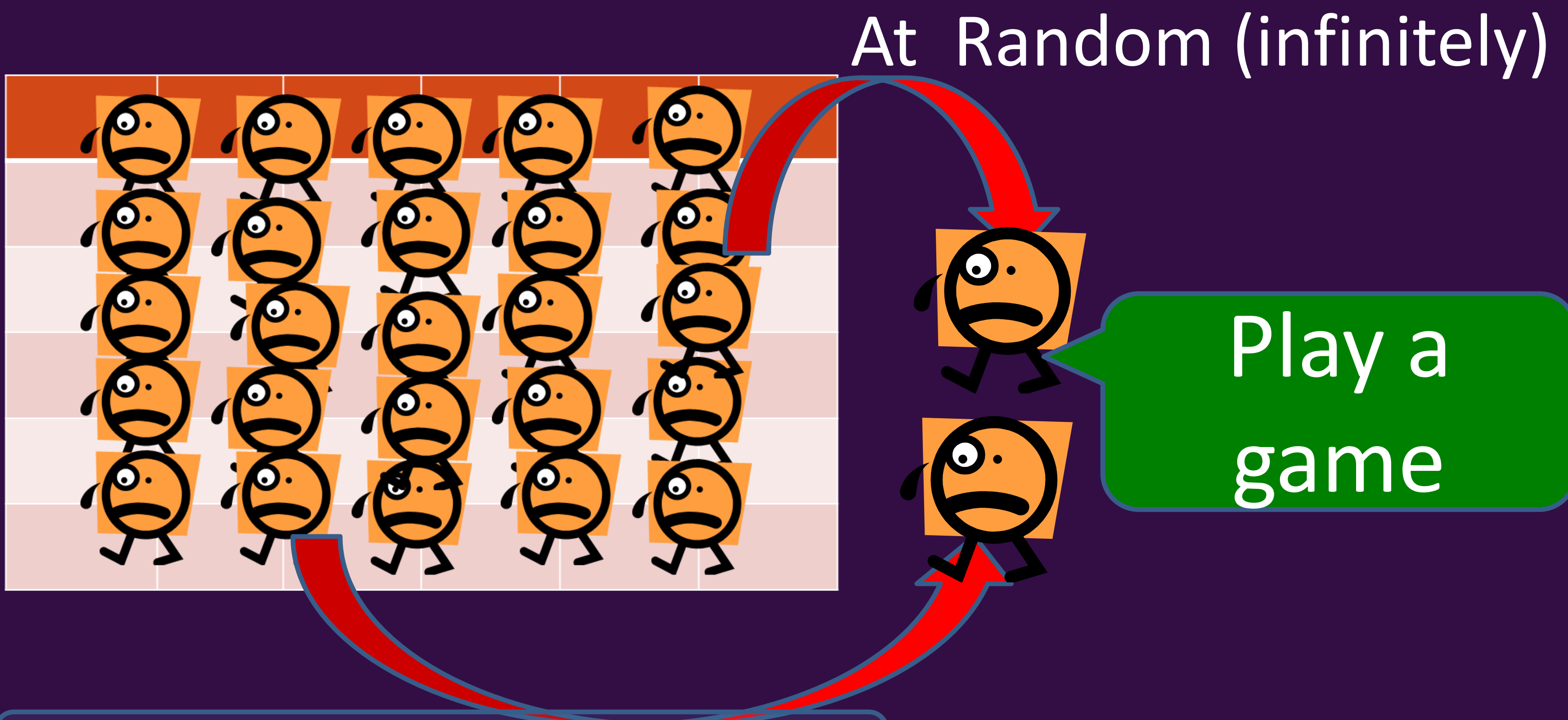


Another players look at the game.

Situation (Traditional Evolutionary Game Theory)



Another players look at the game.

Situation (Traditional Evolutionary Game Theory)

Replicator Equation

REPLICATOR EQ.

$$\dot{x}_i = x_i\left((Ax)_i - x \cdot Ax\right), i = 1, \cdots, n.$$

If the player's payoff from the outcome *i* is greater than the expected utility *x Ax*, the probability of the action *i* is higher than before.

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Two Strategies

$$\dot{x} = x(1-x)\{b-(a+b)x\} \quad \cdots (*)$$

Classification

(I) Non-dilemma: $a > 0, b < 0$, ESS : one

(II) Prisoner's dilemma : $a < 0, b > 0$, ESS :one

(III) Coordination : $a>0, b>0$, ESS two

(IV) Hawk-Dove : $a<0, b<0$, ESS one (mixed strategy)

2

1		S 1	S 2
	S 1	a,a	0,0
	S 2	0,0	b,b

Payoff Matrix

Symmetric and Asymmetic Games

- Difference between Symmetric Two person game and non-symmetric two person game (対称2人ゲームと非対称2人ゲームの違い)

→Player 1 and 2' s payoff matrix are **not** same.

Player 2

Player 1

	S1	S2
S1	A,A	C,B
S2	B,C	D,D

Symmetric Two Person Game

Player 2

Player 1

	S1	S2
S1	A,E	C,G
S2	B,F	D,H

Asymmetric Two Person Game

Replicator Equation： 1本 2本

$$A = A^T$$

$$A \neq A^T$$

Replicator equation and the Lotka-Volterra equation

THEOREM (Hofbauer and Sigmund, 1998, etc.)

There exists a differentiable, invertible map from $\hat{S}_n = \{x \in S_n : x_n > 0\}$ onto R_+^{n-1} mapping the orbits of the replicator equation

$$\dot{x}_i = x_i\left((Ax)_i - x \cdot Ax\right), i = 1, \cdots, n.$$

onto the orbits of the Lotka-Volterra equation

$$\dot{y}_i = y_i\left(r_i + \sum_{j=1}^{n-1} a'_{ij} y_j\right), \qquad i = 1, \ldots, n-1$$

where $r_i = a_{in} - a_{nn}$ and $a'_{ij} = a_{ij} - a_{nj}$

Ultimatum Offer Game
(最終提案ゲーム):

Player：two types (Proposer(提案者)と Responder(応答者))

Situation: These players divide the Π. (プレイヤーの間で, Πの分割を考える)

「提案者が応答者にどのくらいの割合を与えるのか？」

→If the proposer is rational, he proposes the small amount to the responder. (合理的な提案者ならば、できるだけ少なく、応答者に提案する)

→However, it is known that he propose a “fair” trade in economical experiment. (しかし社会経済実験では、「公平な」提案が存在する)（Anomaly）

→Gale, et al.(1995)**[HP]** explains the emergence of the “fair” trade by making the above game’s model with noise.

(このゲームを定式化し, ノイズの影響によって「公平な」提案の発生を説明した)

	Yes	No
Fair	2,2	2,2
Self.	3,1	0,0

$$\dot{y} = (1-\delta_I)y(1-y)(3x-1)+\delta_I\left(\frac{1}{2}-y\right),$$

$$\dot{x} = (1-\delta_{II})x(1-x)(3y-1)+\delta_{II}\left(\frac{1}{2}-x\right)$$

EX：ULTIMATUM OFFER GAME

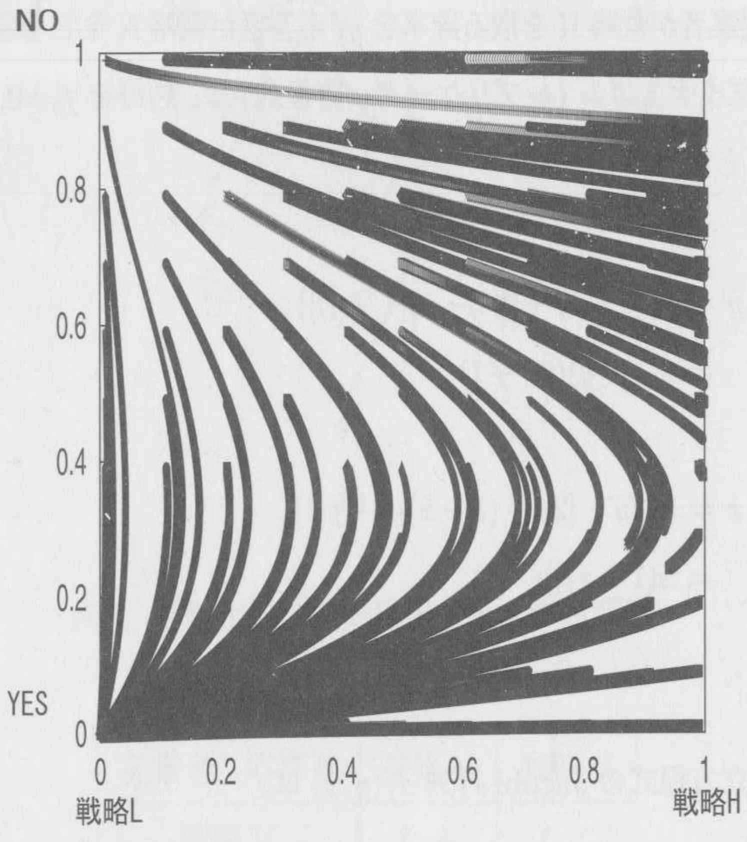

図 2: ノイズがない場合

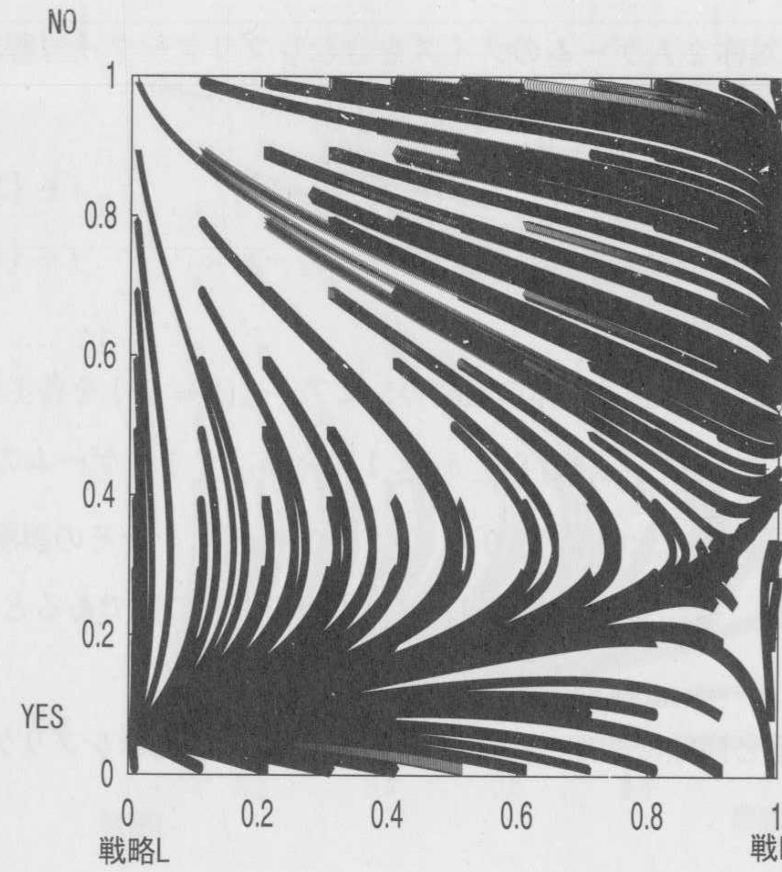

図 3: ノイズがある場合 ($\delta_I = 0.01$, δ_{II}=0.1).

No Noise

Noise

吉川満「非対称2人ゲームの大域的な分析とノイズの役割」『関西学院 経済学研究』第36号 (2005), pp. 21-38. **[HP]**

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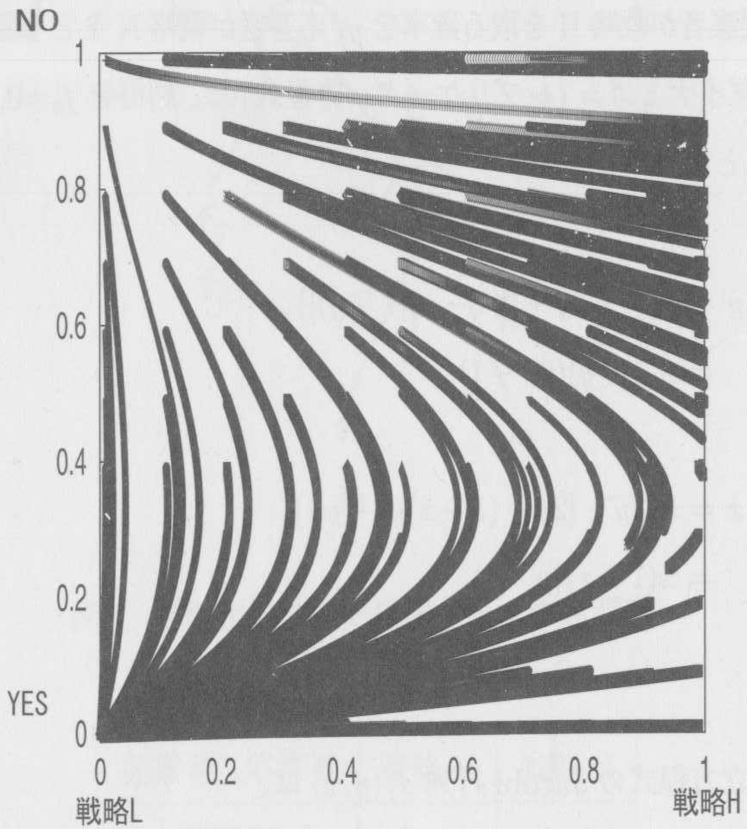

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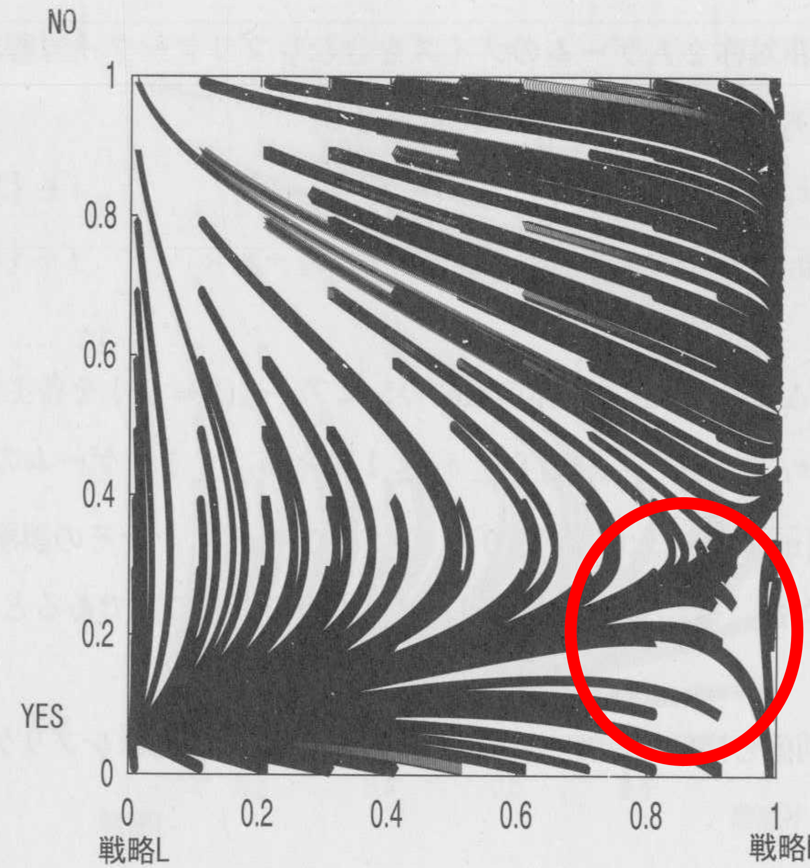

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Stochastic Evolutionary Game Theory (確率的進化ゲーム理論)

- This theory is applied **finite Markov chain**.
- Field : Economics (主に経済学の分野)
- Kandori, et al. (1993) **[HP]**, Young (1993) **[HP]**

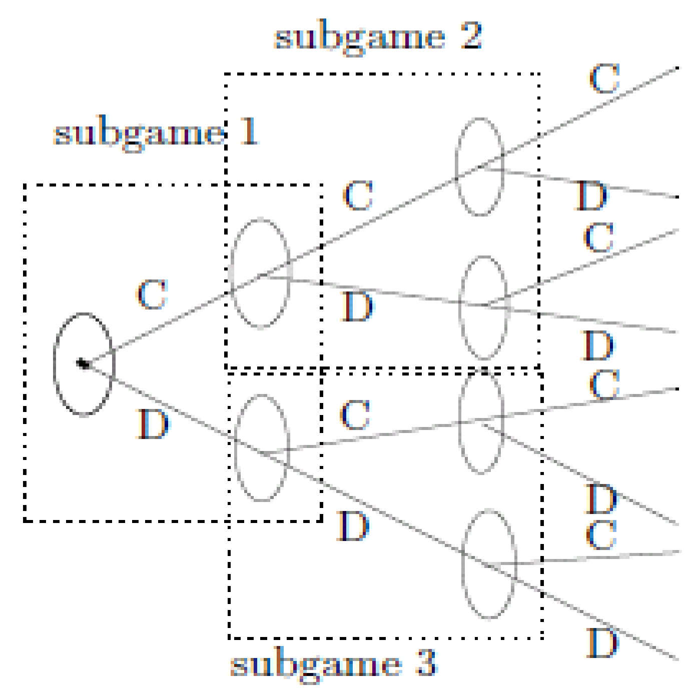


図 2. Markov 連鎖としてのゲーム

We can describe the some phenomena easily.
However, we can only understand the state of equilibrium.

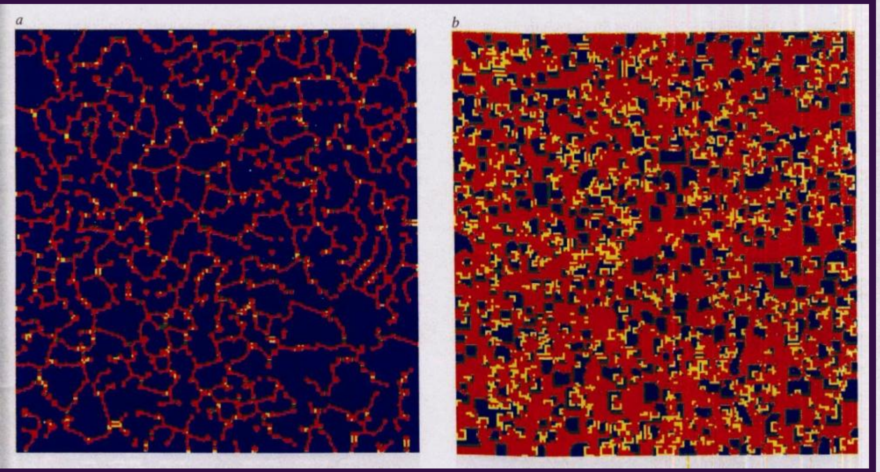
Markov Chain = game in extensive form

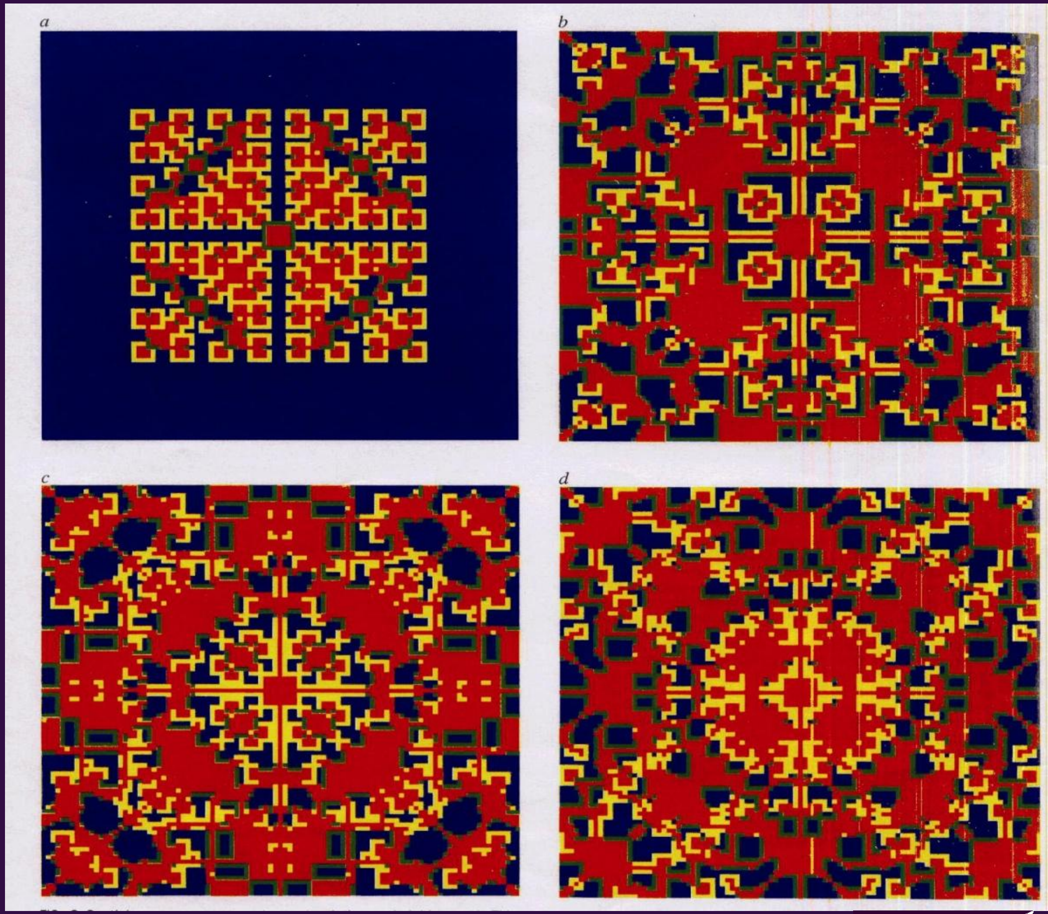
Hot Topic (Mathematics)

SPATIAL STRUCTURE (空間構造)

Basically, by **Simulation**

EX. :SPATIAL PRISONER'S DILEMMA GAME, Nowak and May(Nature, 1992)[HP]





Blue : C(cooperate)

Red : D (defect)

Yellow: D following a C

Green : C following a D

Mathematical Methods

1) **Partial Differential Equation** (偏微分方程式)
 (as **Reaction Diffusion System (反応拡散系として)**)
→ Pure Strategy's Set Space is Infinity
(純粋戦略の集合が無限集合の場合)

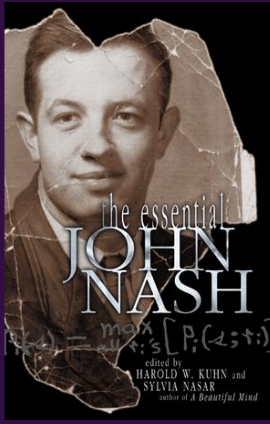
$$\frac{\partial x_i}{\partial t} = D \cdot \nabla^2 x_i + x_i (1 - x_i)\left(a x_i - b(1 - x_i)\right)$$

2) **Statistics Mechanics** (統計力学)
(as Applied **Ising model, Spin Glass**)
→ A lot of games are played simultaneously
(同時に多数のゲームが起こる場合)

Interpretation of Nash Equilibrium (J.F.Nash’s Ph D. Thesis)

- 1. “**Rationality**” ･･･ the players are perceived as rational and they have complete information about the structure of the game, including all of the players’ preferences regarding possible outcomes, where this information about each other’s strategic alternatives and preferences, they can also compute each other’s optimal choice of strategy for each set of expectations. If all of the players expect the same Nash equilibrium, then there are no incentives for anyone to change his strategy.
- 2. “**Statistical Populations**” ･･･ is useful in so-called evolutionary games. This type of game has also been developed in biology in order to understand how the principles of natural selection operate in strategic interaction within among species.(→ **Mass Action**)

(FROM Press Release – The Royal Swedish Academy of Sciences, 11 October 1994)



Nash has received a grant from the National Science Foundation to develop a new “**evolutionary**” solution concept for **cooperative games**.(From *the essential John Nash*)

Summary

- **Evolutionary Game Theory**

1. Describe the out of equilibrium. And evolutionary game theory is equivalent to non-cooperative game in equilibrium.
2. Wide Useful ！(汎用性がある)

→ Assumption ：only one.

- For Beginner,

DEFINE the payoff matrix (利得表を決める)

= **DESCRIBE** the Phenomena.

(現象を記述する)

In progress

Unite ‘’**Time Series Analysis** ’’ and ‘’**Evolutionary Game Theory**’’

→ **Particle Filter ! (粒子フィルタ)**

Thank You For Your Attention

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This File is available at

http://kikkawa.cyber-ninja.jp/

Text Book

For Detail, See my Website(**Bookguide** **Readinglist**)

Classsic:

[1] Maynard Smith, John **Evolution and the Theory of Games** , Cambridge University Press, 1982/10. 日本語訳

[2] Axelrod, Robert **The Evolution of Cooperation** , Basic Books, 1984/03. 日本語訳

Text Book:

[1] Weibull , Jorgen W. **Evolutionary Game Theory** , MIT Press, 1995/08/14. 日本語訳

[2] Hofbauer, Josef and Sigmund, Karl **Evolutionary Games and Population Dynamics** , Cambridge University Press, 1998/07. 日本語訳

[3] Vega-Redondo , Fernando **Evolution, Games and Economic Behaviour** , Oxford University Press, 1997/01.

[4] Samuelson, Larry **Evolutionary Games and Equilibrium Selection (Mit Press Series on Economic Learning and Social Evolution, 1)** , MIT Press, 1997/04.

For Beginner :

[1] 石原英樹, 金井雅之 **進化的意思決定 (シリーズ意思決定の科学)** , 朝倉書店, 2002/04/05.

[2]大浦宏邦 **社会科学者のための進化ゲーム理論―基礎から応用まで** 書房, 2008/09/25.

PRELIMINARIES (EVOLUTIONARY GAME THEORY)

Replicator Eq. and Payoff Matrix

- Strategy : Two, Player : Two
- Payoff

$$P^1 = \begin{pmatrix} f_1 & f_3 \\ f_2 & f_4 \end{pmatrix}, P^2 = \begin{pmatrix} g_1 & g_3 \\ g_2 & g_4 \end{pmatrix}$$

- Replicator Equation

$$\dot{y} = y(1-y)\{f_1 - f_2 + x(f_3 - f_4 - f_1 + f_2)\}$$

$$\dot{x} = x(1-x)\{g_4 - g_2 + y(g_3 - g_4 - g_1 + g_2)\}$$

x is the probability of the type 2 player chooses the strategy 2.
y is the probability of the type 1 player chooses the strategy 1.

$$f_1 - f_2 := a,\; f_4 - f_3 := c,\; g_4 - g_2 := d,\; g_1 - g_3 := b$$

Derive

$$\dot{y} = y(1-y)\{a-(a+c)x\},\; \dot{x} = x(1-x)\{d-(b+d)y\}$$

Classification

(I) Non-Dilemma, **Prisoner's Dilemma :**

ac < 0, bd >0, ESS :1

(II) Coordination :

a>0,b>0,c>0,d>0, ESS :2

(III) Chicken :

a<0,b<0,c<0,d<0, ESS :2

(IV) Matching Pennie:

ab<0,cd<0, ac>0,bd<0, ESS:Mixed

	S1	S2
S1	a,b	0,0
S2	0,0	c,d

- 本研究の一部は，平成20年度採択，文部科学省 グローバルCOEプログラム**「現象数理学の形成と発展」**現象数理若手プロジェクト「人間特有の現象に対する学習の影響 - 進化ゲーム理論による分析 -」に関する研究拠点形成費の助成を受けて行われた.